ON THE INVARIANTS OF AUSLANDER AND MARTSINKOVSKY

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The theory of maximal Cohen–Macaulay (abbreviated to MCM) approximations was established by Auslander and Buchweitz [2], and has played an important role in Cohen–Macaulay representation theory. Moreover, Auslander [1] showed that every finitely generated module M over a commutative Gorenstein local ring admits a unique minimal MCM approximation $0 \to Y_M \to X_M \to M \to 0$, and defined the *$\delta$-invariant* of M as follows.

Definition (Auslander [1])**.** Let R be a commutative Gorenstein local ring and M a finitely generated R-module. Then the *δ-invariant* $\delta_R(M)$ of M is defined as the rank of the largest free summand of the minimal MCM approximation X_M.

Numerous interesting properties and applications of the δ-invariant have been extensively studied; see for example [3, 4, 5, 6, 7, 8, 9, 10, 14, 15, 16]. Later, Martsinkovsky [11, 12] extended the theory of the δ-invariant to the ξ-invariant over general noetherian local rings.

Definition (Martsinkovsky [11, 12])**.** Let R be a commutative noetherian local ring with residue field k and M a finitely generated R-module. Define $\mathrm{V}(M)$ as the subspace of $\mathrm{Hom}_R(M, k)$ consisting of all R-homomorphisms $f : M \to k$ for which there exists a complex homomorphism $\tilde{f}^{\bullet} : P_M^{\bullet} \to P_k^{\bullet}$ with $\tilde{f}^m = 0$ for all $m \gg 0$ and $H^0(\tilde{f}^{\bullet}) = f$, where $P_M^{\bullet}$ and $P_k^{\bullet}$ are minimal free resolutions of M and k, respectively. Then the *ξ-invariant* $\xi_R(M)$ of M is defined as the dimension of $\mathrm{V}(M)$ as a k-vector space.

Martsinkovsky proved that if R is Gorenstein, then the equality $\xi_R(M) = \delta_R(M)$ holds for any finitely generated R-module M. Moreover, by using the ξ-invariant, he gave an extension of the celebrated theorem of Auslander, Buchsbaum and Serre, and partially extended Ding's index theory to general noetherian local rings.

In this talk, we present an approach to analyzing the ξ-invariant using Auslander's approximation theory, based on the preprint [13].

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